IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 04, Issue 04 (April. 2014), ||V4|| PP 23-27

K - Domination Number of Butterfly Graphs

P. Vijaya Jyothi[1], Indrani Kelkar[2], B. Maheswari[3]
1(Department of Mathematics, NBKR Institute of Science and Technology, Vidyanagar, Nellore, India)
2(Department of Mathematics, Acharya Institute of Technology, Bangalore, India)
3(Department of Mathematics, Sri Padmavati Mahila Viswavidyalayam, Tirupati, India)

Abstract: - Butterfly graphs and domination are very important ideas in computer architecture and communication techniques. We present results about one important domination parameter k-domination Number for Butterfly Graph. We find the relation between domination number and k-domination number for BF(n). In this paper we present results about k-domination number of Butterfly Graphs BF(n). We show that domination number and k-domination number of butterfly graphs BF(n) are related to each other as γ_k (BF(n)) > γ(BF(n)) except for k = 2 and n = 6.

Keywords: *Butterfly graph, domination number, k-domination number* ***2000 Mathematics Subject Classification:*** *05C69*

I. INTRODUCTION

Butterfly Networks are interconnection networks which form the back bone of distributed memory parallel architecture. They have very good symmetry in structure and are regular graphs. One of the current interests of researchers is Butterfly graphs, because they are studied as a topology of parallel machine architecture. The vertices model processors and the edges represent communication links between processors.

The vertex set V of BF(n) is the set of ordered pairs (α; v) where $\alpha \in \{0, 1, 2, \ldots\ldots n-1\}$ and v = $x_{n-1}\ x_{n-2} \ldots\ldots x_1\ x_0$ is a binary string of length n where x_i = 0 or 1. There is an edge from a vertex (α; v) to a vertex (α'; v′) where $\alpha' \equiv \alpha + 1 \pmod n$ and $x_j = x_j'\ \forall\ j \neq \alpha'$. A butterfly graph BF(n) is an n - partite graph with n levels. Each level L_k for k = 0, 1, …. , n-1 has 2^n vertices and $L_k = \{ (k; v) / v = x_{n-1}\ x_{n-2} \ldots\ldots x_1\ x_0,\ x_i = 0 \text{ or } 1\}$. Using decimal representation of the binary string we can write $L_k = \{ (k; m) / \text{ where } k = 0, 1, 2, \ldots.\ n-1 \text{ and } m = \sum x_j 2^j, j = 0,1, \ldots.n-1) \}$.

II. MAIN RESULTS

The concept of domination in graph theory was formalized by Berge [2] and Ore [9]. A well known upper bound for the domination number of a graph was given by Ore [9].

A subset D of G is called a k-dominating set of G if every vertex in V \ D is dominated by at least k vertices in D. Cardinality of a minimum k-dominating set is called the k-domination number of G and is denoted by $\gamma_k(G)$.

In this paper we present results on the k-domination number of butterfly graph BF(n). BF(n) is a 4-regular graph, so k-domination is possible only for k < 5 in BF(n). The result for k=2 is proved in [10], here we present the proof for k = 3, 4.

Theorem 1 : The 3-domination number of BF(n) for n > 1 is

$$\gamma_3(\text{BF(n)}) = n\, 2^{n-1} \text{ if } n \leq 4$$
$$= \left\lceil \frac{n}{2} \right\rceil 2^n \quad \text{if } n = 5k \quad k = 1, 2, 3, \ldots..$$

Proof: We give construction of a k-dominating set of BF(n) for n = 2, 3, 4, 5 first and then for n>5 k-dominating sets are found using special Recursive Construction.

Case (i) n = 2

Consider the butterfly graph BF(2). In BF(2) every vertex in one level is adjacent to 3 vertices in the other level. Let D denote a 3-dominating set of BF(2). For a vertex to be 3-dominated, at least 3 vertices adjacent with that vertex are needed to be in the 3-dominating set. So $|D| \geq 3$.

Let v=(1, t) be a vertex from L_1, v has three vertices adjacent in L_0. To 3-dominate v, all these vertices must be included into D. Now there is one vertex left in L_0 including it in D makes each vertex of L_1 to be 3-dominated giving $|D|$ = 4, This is the minimum cardinality as any other 3-dominating set will include vertices from two levels and so will have more than four vertices.

So γ_3(BF(2)) = 4 = 2 x 2^{2-1} = n x 2^{n-1}.

Case (ii) n = 3.

Consider butterfly graph BF(3). We know that BF(3) has three levels L_0, L_1 and L_2 with 8 vertices each and every vertex in a level L_k is adjacent to two vertices in the preceding level L_{k-1} and two vertices in the succeeding level L_{k+1}. So for 3-domination of a vertex, two vertices from one level and one vertex from other level must be selected into D

Let D denote a 3-dominating set of BF(3). Consider a vertex from level L_1 $(1; m_1)$ in the left copy of BF(3), that is $m_1 \varepsilon \{0,1,2,3\}$. To 3-dominate this vertex, three adjacent vertices of $(1; m_1)$ must be included into D. Vertex $(1; m_1)$ is adjacent to two vertices in L_0, $(0; m_1)$, $(0; m_2)$, with $m_1 \neq m_2$ and $|m_1 - m_2| = 2$, which belong to $K_{2,2}$ between L_0 - L_1 and two vertices in L_2 viz. $(2; m_1)$, $(2; s_1)$, with $m_1 \neq s_1$ and $|m_1 - s_1| = 2^2$ where $s_1 \varepsilon \{4, 5, 6, 7\}$, which belong to a $K_{2,2}$ between L_1 and L_2.

Without loss of generality, take two vertices from L_0 and one vertex from L_2 into D giving $D = \{(0; m_1), (0; m_2), (2; s_1)\}$, where $s_1 = m_1$ or $|m_1 - s_1| = 2^2$. Vertices of D 3-dominate $(1; m_1)$. Next Consider vertex $(1; m_2)$ which is dominated by $(0; m_1)$ and $(0; m_2)$. To 3-dominate this vertex, one of its adjacent vertices $(2; s_2)$ with $s_2 = m_2$ or $|m_2 - s_2| = 2^2$ from L_2 is to be included where $s_1 \neq s_2$.

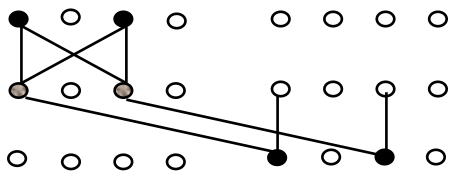

Figure 1

The vertices $\{(0; m_1), (0; m_2), (2; s_1), (2; s_2)\}$ can 3-dominate only two vertices $(1, m_1)$ and $(1, m_2)$. None of these four vertices are adjacent to other two vertices from left copy of L_1. As shown in figure 1, to dominate (1;0) and (1;2) we need four vertices (0;0), (0;2), (2;4), (2;6). So to 3-dominate remaining vertices of level L_1 in the left copy we need to include four more vertices to D.

Giving $D = \{(0; m_1), (0; m_2), (2; s_1), (2; s_2), (0; m_3), (0; m_4) (2; s_3), (2; s_4)\}$

Now the vertices of L_1 in the left copy of BF(3) are 3-dominated by the vertices in D. Similarly to 3-dominate vertices of L_1 in the right copy, 4 remaining vertices each from levels L_0 and L_2 need to be included to D. But the vertices of D 2-dominate remaining vertices from L_0 and L_2 so if we include $(1; s_1)$, $(1; s_2)$, $(1; s_3)$, $(1; s_4)$ into D then all the vertices of V\D are 3-dominated. Therefore the minimum cardinality of a 3-dominating set is 12.

Two possible minimal 3-dominating sets of BF(3) are

$D_1 = \{(0;0), (0;1), (0;2), (0;3), (1;4), (1;5), (1;6), (1;7), (2;4), (2;5), (2;6), (2;7)\}$

$D_2 = V \setminus D_1$, Complement of D_1. Hence $\gamma_3(BF(3)) = 12 = 3 \times 2^{3-1} = n \times 2^{n-1}$.

Case (iii) n = 4

Consider BF(4). As in case (ii), to 3-dominate vertices in level L_1 we select 16 vertices from L_0 and 8 vertices from L_2. These vertices 3-dominate all vertices of level L_3 also.

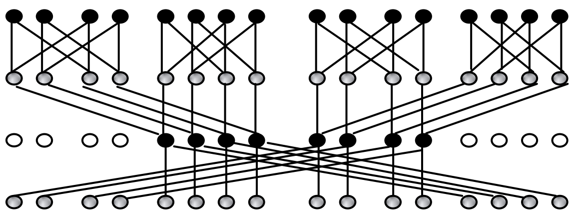

Figure 2

Now to 3-dominate remaining vertices of L_2 we have two choices, either select vertices from two levels L_1 and L_3 or include these 8 vertices into 3-dominating set. For minimal number of vertices in 3-dominating set we include these 8 vertices of L_3 into 3-dominating set. Therefore all vertices from L_1 and L_3 form a 3-dominating set of BF(4). Similarly set of all vertices from L_0 and L_2 form another 3-dominating set of BF(4) of same cardinality 2×2^4.

Therefore $\gamma_3(BF(4)) = 2 \times 2^4 = 4 \times 2^{4-1} = n2^{n-1}$ □

Case (iv) n = 5.

Consider the butterfly graph BF(5). By the Recursive Construction given in [8], BF(5) has 2 copies of BF(4) and a new level L_4 with 2^5 vertices. Let D_1 and D_2 denote 3-dominating sets in the left and right copy of BF(4) respectively. Since γ_3(BF(4)) = 32, select 32 vertices from left copy of BF(5) into D_1 and 32 vertices from right copy of BF(5) into D_2. Let D_1 = {(0; i), (2; i) / i = 0 to15} . Then all the vertices in left copy of L_1 are 3-dominated by the vertices in D_1 and the vertices (3; i), i = 0....15 are 2-dominated. To 3-dominate these vertices there are two choices. One choice is to select all vertices of L_4 and another choice is to select all vertices of L_3. If the vertices of L_4 are selected, then these vertices will 1-dominate the vertices of L_3 which are already 2-dominated. So they will be 3-dominated. Similar is the case with the selection of vertices of $L_{3.}$ Hence the vertices of L_4 are taken into D_1 so that D_1 = {(0; m_i), (2; m_i), (4; m_i) / i = 0..15 } and $|D_1| = 3.\ 2^4$.

Using the symmetry of butterfly graphs, mirror image compliment set of D_1 in the right copy of BF(5) will 3-dominate all vertices in that copy and it is given by

D_2 = {(0; m_i), (2; m_i), (4; m_i) / i = 16…..31}.

Hence D = $D_1 \cup D_2$ becomes a 3-dominating set of BF(5) with $|D| = 3.2^5 = 96$.

We claim that D is a minimum 3-dominating set of BF(5). If possible let there be a 3-dominating set S such that │S│<│D. As the domination number of BF(n) is even, due to the symmetry of the graph, │S│ is even. Now │S│<│D| implies │S│= 94 or │S│< 94. Suppose │S│= 94. Again by symmetry, 47 vertices in the left copy and 47 vertices in the right copy are to be selected. But this is not a choice as domination number is even.

Suppose 46 vertices in the left copy and 48 vertices in the right copy are selected. Then as above 48 vertices will 3-dominate all vertices in the right copy of BF(3). Out of 46 vertices in the left copy, as γ_3(BF(4)) = 2^5, balance 14 vertices may 2-dominate (it is also not possible) the remaining 16 vertices in BF(5), but can not 3-dominate, because to 3-dominate a vertex we require at least 3 vertices. Similar is the case if │S│ < 94. Hence D is a minimum 3-dominating set of BF(5). Thus γ_3(BF(5)) $= 3.2^5 = = \left\lceil \frac{n}{2} \right\rceil 2^n$ □

One such selection of 3-dominating set for BF(5) is illustrated in the following figure.

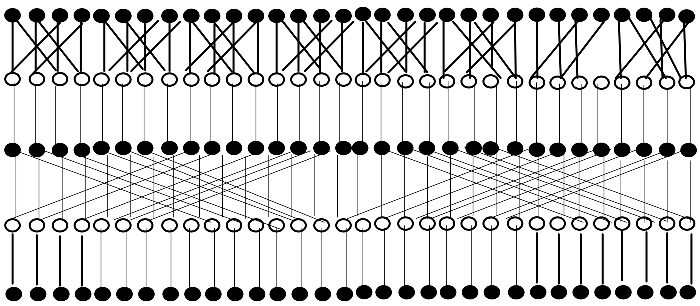

Figure 3

Case (v) n = 5k

The result is proved by using the Principle of Mathematical Induction on k.

Step 1: Let k = 1. So n = 5. From case (iv) the result is true for BF(5).

Step 2: Let us assume that the result is true for k = t. Then the result is proved for k = t + 1. Consider the graph BF(5(t+1)). From Recursive Construction 2 [8], BF(5t+5) is decomposed into 2^5 copies of BF(5t) without wings and the last 5 levels form a pattern of BF(5) with 2^5 vertex groups where each vertex group has 2^{5t} vertices.

From the induction hypothesis the result is true for BF(5t). So consider a 3-dominating set S_i of cardinality $\left\lceil \frac{5t}{2} \right\rceil 2^{5t}$ for each copy of BF(5t). In BF(5t+5) the last 5 levels form a pattern of BF(5), which is isomorphic to BF(5) without wings. The choice of vertices into a 3-dominating set for BF(5), is from levels L_0, L_2 and L_4. Hence this result obtained for the graph BF(5) can be extended to a pattern of BF(5).

Thus a 3-dominating set D_1 is obtained with 32 vertex groups (recall that each vertex group is treated as one vertex) in L_0, L_2 and L_4 of the last 5 levels, where each vertex group has 2^{5t} vertices.

Suppose a 3-dominating set of cardinality less than $32x3x2^{5t}$ for the pattern of BF(5) is obtained. Since the pattern of BF(5) is isomorphic to the graph BF(5) without wings, this assumption gives the existence of a 3-dominating set of cardinality less than 3.2^5 for BF(5), which is a contradiction as γ_3(BF(5)) = $3x2^5$.

Hence D_1 is a minimum 3-dominating set for a pattern of BF(5). Let $D = \bigcup_{i=1}^{32} S_i \cup D_1$.

As S_i are minimum 3-dominating sets of disjoint copies of BF(5t) between levels L_0 to L_{5t-1} and D_1 is a minimum 3-dominating set for pattern of BF(5) between levels L_{5t} to L_{5t+4} , it follows that D is a disjoint union of minimum 3-dominating sets.

Hence D is a minimum 3-dominating set of BF(5t+5) whose cardinality is

$$|D| = 32.\left\lceil\frac{5t}{2}\right\rceil 2^{5t} + 32.3.\ 2^{5t}.$$

$$= \left\lceil\frac{5t}{2}\right\rceil.\ 2^{5t+5} + \left\lceil\frac{5}{2}\right\rceil 2^{5t+5}.$$

$$= \left\lceil\frac{5(t+1)}{2}\right\rceil 2^{5(t+1)}.$$

So the result is true for k = t + 1.

Hence by the Principle of Mathematical Induction the result is true for all positive integers k.

Thus $\gamma_3(BF(5k)) = \left\lceil\frac{5k}{2}\right\rceil \mathbf{2^{5k}} = \left\lceil\frac{n}{2}\right\rceil \mathbf{2^{n}}$ □

Corollary 1 : $\gamma_3(BF(5k+1)) = \left\lceil\frac{5k+1}{2}\right\rceil \mathbf{2^{5k}} = \left\lceil\frac{n}{2}\right\rceil \mathbf{2^{n}}$

Proof : In BF(5k+1), from each of two consecutive levels we need to include vertices from one level into 3-dominating set for example L_0, L_2, L_4, ...L_{n-2} together 3-dominate all vertices of the other levels giving minimal 3-dominating set with cardinality $\left\lceil\frac{n}{2}\right\rceil 2^n$**.** One such selection of 3-dominating set for BF(6) is illustrated in the following figure, where dominating set has all vertices from levels L_0, L_2 and L_4.

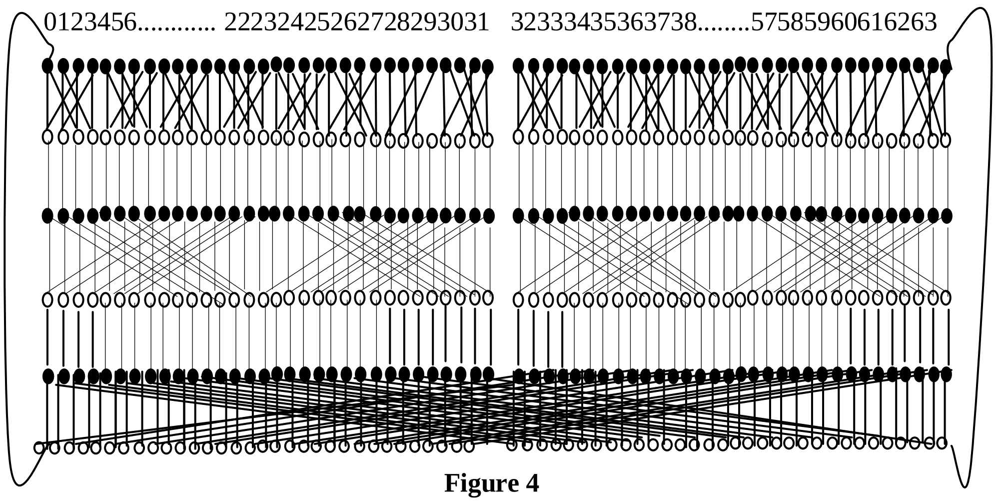

Figure 4

Corollary 2 : $\gamma_3(BF(n)) = \left\lceil\frac{n}{2}\right\rceil \mathbf{2^{n}}$ for n=5k+2, 5k+3 and 5k+4 .

Theorem 2 : The 4-domination number of BF(n) for n > 2 is $\gamma_4(BF(n)) = \left\lceil\frac{n}{2}\right\rceil 2^n$

Proof : By definition of edges in BF(2), every vertex is adjacent to three vertices only with a straight and winged edge between (0, t) and (1, t) for t = 0, 1, 2, 3 which are parallel edges. So for BF(2) 4-domination is not possible giving $\gamma_4(BF(2)) = 0$.

In BF(n), n > 2, every vertex is of degree 4, all the 4 vertices must be included in 4-dominating set. As a vertex from level L_r is adjacent to two vertices each from L_{r-1} and L_{r+1} (modulo n), to 4-dominate all vertices of L_r all vertices from L_{r-1} and L_{r+1} must be included in to 4-dominating set. So $\left\lceil\frac{n}{2}\right\rceil$ levels all 2^n vertices should be in to 4-dominating set which will be the minimal 4-dominating set. Therefore $\gamma_4(BF(n)) = \left\lceil\frac{n}{2}\right\rceil 2^n$.

Observations: 1. $\gamma_3(BF(n)) = \gamma_4(BF(n))$ for $n \neq 3$ as every 3-dominating set is a 4-dominating set.

2. From [8], $\gamma(BF(n)) = \left\lceil\frac{n}{2}\right\rceil 2^{n-1}$ for n=2, 4k, 4k+1, 4k+3

$= (\left\lceil\frac{n}{2}\right\rceil + 1)2^{n-1}$ for n= 3, 4k+2

From [10] $\gamma_2(BF(n)) = k\ 2^n + r\ 2^{n-1}$ for n = 3k+r

$\gamma_3(BF(n))$, $\gamma_4(BF(n))$ found above :

The following table gives domination and k-domination number of various butterfly graphs

Graph	γ(BF(n))	γ_2(BF(n))	γ_3(BF(n))	γ_4(BF(n))
BF(2)	2	4	4	4
BF(3)	6	8	12	16
BF(4)	16	24	32	32
BF(5)	48	64	96	96
BF(6)	128	128	192	192
BF(7)	256	320	512	512
BF(n) n > 4	$\left\lceil \frac{n}{2} \right\rceil 2^{n-1}$	$(2k + r)\ 2^{n-1}$	$\left\lceil \frac{n}{2} \right\rceil 2^{n}$	$\left\lceil \frac{n}{2} \right\rceil 2^{n}$

From above table it is evident that $\gamma_k(BF(n)) > \gamma(BF(n))$ for $k = 2, 3, 4$ and $n \neq 6$.

REFERENCES

[1] Barth D. and Raspaud A, Two edge - disjoint Hamiltonian Cycles in the Butterfly Graphs, Information processing letters, 51(1994), 175–179.

[2] Berge C., Theory of graphs and its Applications, Methuen, London, 1962.

[3] Berge C., Graphs and Hypergraphs, North Holland Amsterdam, 1973.

[4] Cockayne E J, Gamble B, Shephard B, An upper bound for k-domination number of a graph, Journal of Graph Theory, 2006, Vol 9 Issue 4, pp 533-534.

[5] Fink J.F., Jacobson M.S., n-Domination in Graphs, in: Alavi Y. and Schwenk A. J.(eds), Graph Theory with Applications to Algorithms and Computer Science, New York, Wiley, 1985, 283–300.

[6] Harary F., Graph Theory, Addison-Wesley, Reading, Mass, 1972.

[7] Haynes T W, Hedetniemi S T and Slater P J, Fundamental of Domination in Graphs: Advanced Topics, (Marcel Dekker, Inc, New York,

[8] Indrani P. Kelkar, Some Studies On Domination Parameters of Butterfly Graphs, Ph.D Thesis, Sri Padmavathi Mahila Visvavidyalayam, Tirupati India, 2006.

[9] O. Ore, Theory of Graphs, Amar. Maths. Soc. Colloq. Pub., 38, Probidence, 1962.

[10] Vijayajyothi P, Indrani Kelkar, B Maheswari, 2-domination number of BF(n), Chamchuri Journal of Mathematics, Vol 1, No. 1, pp 73-80, 2008

[11] Volkman Lutz, A bound on k-domination number of graph, Czekoslovak Mathematical Journal, March 2010, Vol 60, Issue 1, pp77-83